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Simple model of a limit order-driven market

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Abstract

We introduce and study a simple model of a limit order-driven market. Traders in this model can either trade stock (or any other risky asset for that matter) at the market price or place a limit order, i.e., an instruction to buy (sell) a certain amount of the stock if its price falls below (raises above) a predefined level. The choice between these two options is purely random (there are no strategies involved), and the execution price of a limit order is determined simply by offsetting the most recent market price by a random amount. Numerical simulations of this model revealed that despite such minimalistic rules the price pattern generated by this model has such realistic features as "fat" tails of the probability distribution of price fluctuations, characterized by a crossover between two power law exponents, long range correlations of the volatility, and a non-trivial Hurst exponent of the price signal.

In recent years a considerable effort was invested in high-quality statistical analysis of short-time fluctuations in prices of various risky assets. [1] The unifying feature of all these assets is that their price is determined by law of supply and demand while the asset being traded on an open market. These studies resulted in a discovery of robust and to a certain degree universal features of such fluctuations, and triggered theoretical studies aimed at explaining or simply mimicking these observations. The list of empirical facts that need to be addressed by any successful theory or model is:

(i) The histogram of short time-lag increments of market price has a very peculiar non-Gaussian shape with a sharp maximum and broad wings [3]. The current consensus about the functional form of this distribution is that up to a certain point it follows a Pareto–Levy distribution, with the exponent of its power law

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[1] A general discussion of the phenomenology of market prices can be found in two recent books by Bouchaud and Potters [1] (a draft of its English translation can be downloaded from http://www.science-finance.fr/book.html), and Mantegna and Stanley [2].

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tail $1 + \alpha_1 \sim 2.4 - 2.7$, after which it crosses over either to a steeper power law with an exponent $1 + \alpha_2 \sim 3.7 - 4.3$ [4–6], or, as was reported in earlier studies [7,8], to an exponential decay. In both cases this crossover ensures a finite variance (second moment) of the distribution.

(ii) When viewed on time scales less than several trading days, the graph of price vs. time appears to have a Hurst exponent $H \simeq 0.6$–0.7 [3,8], different from an ordinary uncorrelated random walk value $H_{RW} = 0.5$.

(iii) The volatility (the second moment of price fluctuations) exhibits correlated behavior. It is manifested in clustering of volatility, i.e., the presence of regions of unusually high amplitude of fluctuations separated by relatively quiet intervals, visible with a "naked eye" in the graph of price increment vs. time. These clustering effects determine the shape of the autocorrelation function of volatility as a function of time, which was shown to decay as a power law with a very small exponent $\gamma \simeq 0.3$–0.4 and no apparent cutoff [7,9].

There are several approaches to modeling market mechanics. In one type of models price fluctuations result from trading activity of conscious agents, whose decisions to buy or sell are dictated by strategies they follow. These strategies evolve in time (often according to some Darwinian rules) and give rise to a slowly changing fluctuation pattern. There is little doubt that the evolution and dynamics of investor's strategies and beliefs influence the long term behavior of real market prices. For example, if some company could not keep up with the competition, sooner or later investors would realize it, and in the long-term its stock price would go down. However, it is unclear how does it influence the properties of stock price fluctuations at very short timescales, which do not allow time for traders to update their strategies or for a company to change its profile. Another problem with models explaining short time price fluctuations in terms of strategy evolution is that they inevitably lead their creators to shaky grounds of speculations about relevant and irrelevant psychological motivations of a "typical" trader in a highly heterogeneous trader population. The remarkable universality of the general features of price fluctuations in markets of different types of risky assets such as stocks and their derivatives, foreign currencies, and commodities (say, cotton or oil) makes one to suspect that in fact psychological factors play little role in determining their short time properties, and leads one to try to look for a simpler mechanism giving rise to these features.

In this work we do a first step in this direction by introducing and numerically studying a simple market model, where a nontrivial price pattern arises not due to the evolution of trading strategies, but rather as a consequence of *trading rules themselves*, and to the way in which supply and demand determine the price on an open market. Before we proceed with formulating the rules of our model we need to define several common market terms. A trader on an open market is usually allowed to place a so-called "limit order to sell (buy)", which is an instruction to automatically sell (buy) a particular amount of the traded asset, which we for simplicity will be simply referring to as stock, if its market price would raise higher (or drop lower for a limit buy order) than the predetermined threshold. This threshold is sometimes referred to as the execution price of the limit order. In many modern markets, known to economists as order-driven markets,[2] limit orders placed by ordinary traders constitute the major source of the liquidity of the market. It means that a request to *immediately* buy or sell a particular amount of stock at the best market price, or "market order", is filled by matching it to the best unsatisfied limit order in the limit order book. To illustrate how orders are executed in an order-driven market let us consider the following simple example: suppose one trader (trader #1) has submitted a limit order to sell 1000 shares of the stock of a company X, provided that its price would exceed \$20/share. Subsequently, another trader (trader #2) has submitted a limit order to sell 2000 shares of X if the price would exceed \$21/share. Finally, a third trader decides to buy 2000 shares of X at the market price. In the absence of other limit orders his order will be filled as follows: he will buy 1000 shares from trader #1 at \$20/share and 1000 shares from trader #2 at \$21/share. After this transaction the limit order book would contain only one partially filled limit order to sell, that of trader #2 to sell 1000 shares of X at \$21/share.

Traders in our model can either trade stock *at the market price* or place a *limit order to sell or buy*. To simplify the rules of our toy market, traders are allowed to trade only one unit (lot) of stock in each transaction. That makes all limit and market orders to be of the same size. The empirical study of limit-order trading at the ASX[2] can be used to partially justify this simplification. In this work it was observed that limit orders mostly come in sizes given by round numbers such as 1000 shares and (to a lesser extent) 10 000 shares. Unlike many other market models, we do not fix the number of traders. Instead, at each time step a "new" trader appears out of a "trader pool" and attempts to make a transaction. *With equal probabilities this new trader is a seller or a buyer*. He then performs one of the following two actions:

- with probability q_{lo} he places a limit order to sell/buy.
- otherwise (with probability $1 - q_{lo}$) he trades (sells or buys) at the market price.

The rule of execution of a market order in our model is particularly simple. Since all orders are of the same size, a market order is simply filled with the best limit order (i.e., the highest bid among limit orders to buy and the lowest ask among limit orders to sell), which is subsequently removed from the limit order book. This transaction performed at the execution price of the best limit order sets a new value of the market price $p(t)$.

To complete the definition of the rules one needs to specify how a trader who selected to place a new limit order decides on its execution price. Traders in our model do this in a very "non-strategic" way by simply offsetting the price of the last transaction performed on the market (current market price $p(t)$), by a random number Δ. This *positive* random number is drawn each time from the same probability distribution $P(\Delta)$. A new limit order to sell is placed *above the current price* at $p(t) + \Delta$, while a

[2] A nice introduction to the mechanics of limit-order driven markets in general, and the Australian Stock Exchange in particular, by W. Yang can be found at http://www.af.ecel.uwa.edu.au/accfin/WorkingPapers/abstracts/99-98.htm.

new limit order to buy – below it at $p(t)-\Delta$. This way ranges of limit orders to sell and to buy never overlap, i.e., there is always a positive gap between highest bid and lowest ask prices. This "random offset" rule constitutes a reasonable first-order approximation to what may happen in real order-driven markets and is open to modifications if it fails a reality check. The most obvious variants of this rule, which we plan to study in the near future, are (i) A model where each trader has his individual distribution $P(\Delta)$. This modification would allow for the coexistence of "patient" traders who do not care very much about when their order will be executed or if it will be executed at all, and can therefore select large Δ and pocket the difference, and "impatient" traders who need their order to be executed soon, so they tend to select a small Δ or trade at the market price. (ii) A model in which the probability distribution of Δ is determined by the historic volatility of the market. This rule seems to be particularly reasonable description of a real order-driven market. Indeed, if traders selection of Δ is influenced primarily by his desire to reduce waiting time before his order is executed, then it would make sense to select a larger Δ in a more volatile market, which is likely to cover larger price interval during the same time interval. However, before any of these more complicated versions of this rule could be explored one needs to study and understand the behavior of the base model, where Δ is just a random number, uncorrelated with volatility and/or the individual trader profile.

One should notice that the behavior of traders in our model is completely passive and "mechanical": once a limit order is placed it cannot be removed or shifted in response to a current market situation. This makes our rules fundamentally different from these of the Bak–Paczuski–Shubik (BPS) model [10], where traders randomly increase or decrease their quotes at each time step. Such haphazard trader behavior cannot be realized in an order-driven market, where each change of the limit-order execution price carries a fee.

We have simulated our model with $q_{lo}=\frac{1}{2}$, i.e., when on average half of the traders select to place limit orders, while the other half trade at the market price. The random number Δ, used in setting an execution price of a new limit order, was drawn from a uniform distribution in the interval $0\leqslant\Delta\leqslant\Delta_{\max}=4$. Obviously, price patterns in models with different values of $\Delta_{\max}$ are identical up to an overall rescaling factor. Our choice of $\Delta_{\max}=4$ was dictated by the desire to compare the behavior of the model with continuous spectrum of Δ to that with a discrete spectrum $\Delta=\{1,2,3,4\}$. Discrete spectrum of Δ may better compare to the behavior of real markets, where all prices are multiples of a unit tick size. Our comparison confirmed that most scaling properties of the price pattern are the same in both variants. We were surprised to notice that non-trivial features of our model survived even in a model with deterministic $\Delta=1$. To improve the speed of numerical simulations we studied a version of the model, where only 2^{17} lowest ask and highest bid quotes were retained. The list of quotes was kept ordered at all times, which accelerated the search for the highest bid and lowest ask limit orders whenever a transaction at the market price was requested. We also studied a variant of our model where each limit order had an expiration time: if a limit order was not filled within 1000 time steps it was removed from the list. Not only this rule

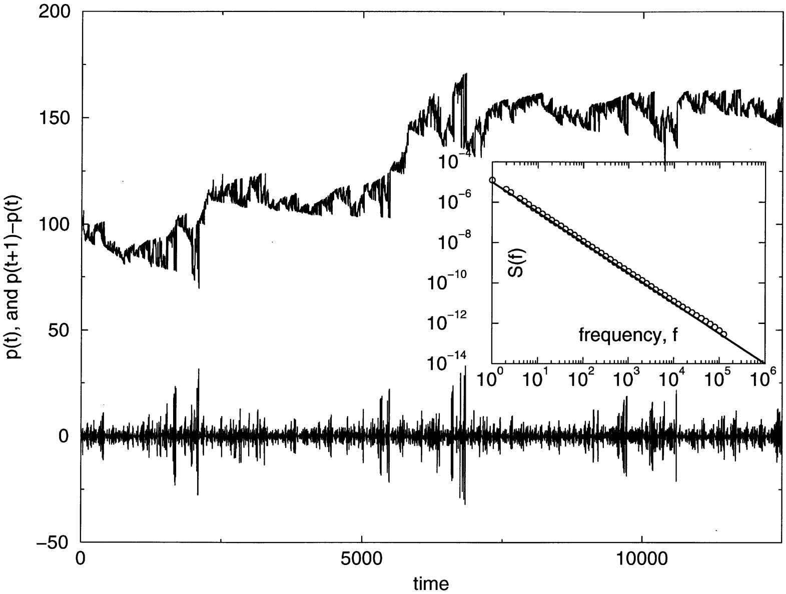

Fig. 1. The price signal $p(t)$ and its derivative $\delta p = p(t+1) - p(t)$ as a function of time. The inset shows the Fourier transform of the autocorrelation function of the price signal. Solid line is a fit $S_p(f) = f^{-(1+2H)} = f^{-3/2}$, corresponding to the Hurst exponent $H = \frac{1}{4}$.

prevented an occasional accumulation of a very long list of limit orders, but also it made sense in terms of how limit orders are organized in a real market. Indeed, limit orders at, for example, New York Stock Exchange are usually valid only during the trading day when they were submitted. There are also so called "good till canceled" (or open) orders, which are valid until they are executed or withdrawn. [3] Then the version of our model, where the expiration time of a limit order is not specified corresponds to all orders being "good till canceled", while the version, where only the most recent orders are kept, mimics the market composed of only "day orders". We have checked that for any reasonably large value of the cutoff parameter, no matter if it is an expiration time or the number of best sell/buy orders to keep, one ends up with the same scaling properties of price fluctuations.

In Fig. 1, we present an example of price history in one of the runs of our model. Visually it is clear that this graph is quite different from an ordinary random walk. This impression is confirmed by looking at the pattern of price increments $p(t+1) - p(t)$, shown in the same figure. One can see that large increments are clustered in regions of high volatility, separated by relatively quite intervals. The Fourier spectrum of the price signal averaged over many runs of the models provides us with a value of the Hurst exponent H of the price graph. Indeed, the exponent of the Fourier transform of price autocorrelation function $S_p(f)$ is related to the Hurst exponent as $S_p(f) \sim$

[3] The definitions of key market terms used at the NYSE can be found by selecting "Glossary" at http://www.nyse.com/marketinfo/marketinfo.html.

$f^{-(1+2H)}$. The log–log plot of $S_p(f)$, logarithmically binned and averaged over multiple realizations of the price signal of length 2^{18}, is shown in the inset to Fig. 1. It has an exceptionally clean $f^{-3/2}$ functional form for over 5 decades in f, which corresponds to the Hurst exponent of the price signal $H = \frac{1}{4}$. This exponent is definitely different from its random walk value $H_{\mathrm{RW}} = \frac{1}{2}$. A Hurst exponent $H = \frac{1}{4}$ was also observed in the Bak–Paczuski–Shubik model A [10,11]. An intuitive argument in favor of a small Hurst exponent can be constructed for our model. According to the rules of the model an execution price of a new limit order is always determined relative to the current price. It is also clear that a large density of limit orders around current price position reduces its mobility. Indeed, in order for the price to move to the new position all limit orders in the interval between the current and new values of the price must be filled by market orders. If for one reason or the other the price remained fairly constant for a prolonged period of time, limit orders created during this time tend to further trap the price in this region. This self-reinforcing mechanism qualitatively explains the slow rate of price change in our model. Unfortunately, the nontrivial Hurst exponent $H = \frac{1}{4}$ is a step in the wrong direction from its random walk value $H_{\mathrm{RW}} = \frac{1}{2}$. Indeed, the short time Hurst exponent of real stock prices was measured to be $H_{\mathrm{real}} \simeq 0.6 - 0.7$.

The amplitude of price fluctuations in our model has significant long-range correlations. One natural measure of these correlations is the autocorrelation function of the *absolute value* of price increments $S_{abs}(t) = \langle |p(t'+t+1) - p(t'+t)||p(t'+1) - p(t')|\rangle_{t'}$ [9]. In our model this quantity was measured to have a power-law tail $S_{abs}(t) \propto t^{-1/2}$. This is illustrated in Fig. 2 where the Fourier transform of $S_{abs}(t)$ has a clear $f^{-1/2}$ form. The exponent $\gamma = \frac{1}{2}$ of $S_{abs}(t) \propto t^{-\gamma}$ in our model is not far from $\gamma = 0.3$ measured in the S&P 500 stock index [9]. In Fig. 2 we also show the Fourier transform of the autocorrelation function of *signs* of price increments $S_{sign}(t) = \langle sign[p(t' + t + 1) - p(t' + t)] sign[p(t' + 1) - p(t')]\rangle_{t'}$, which has a white noise (frequency independent) form. This is again, similar to the situation in real market, where signs of price increments are known to have only short-range (<30 min) correlations.

Finally, in Fig. 3 we present three histograms of price increments $p(t + \delta t) - p(t)$ in our model, measured with time lags $\delta t = 1, 10, 100$. The overall form of these histograms is strongly non-Gaussian and is reminiscent of the shape of such distribution for real stock prices. As the time lag is increased the sharp maximum of the distribution gradually softens, while its wings remain strongly non-Gaussian. In the inset we show a log–log plot of the histogram of $p(t+1) - p(t)$ ($\delta t = 1$) collected during $t_{\mathrm{stat}} = 3.5 \times 10^7$ timesteps (as compared to $t_{\mathrm{stat}} = 40\,000$ for the data shown in the main panel) and logarithmically binned. One can clearly distinguish two power law regions separated by a sharp crossover around $p(t + 1) - p(t) \simeq 1$. The exponents of these two regions were measured to be $1 + \alpha_1 = 0.6 \pm 0.1$ and $1 + \alpha_2 = 3 \pm 0.2$. The power-law exponent $1 + \alpha_2 = 3$ of the far tail lies right at the borderline, separating the Pareto–Levy region $1 + \alpha < 3$, where the distribution has an infinite second moment, from the Gaussian region. In any case, since price fluctuations in our model were shown to have long-range correlations, one should not expect convergence of the price fluctuations distribution to a universal Pareto–Levy or Gaussian functional form as δt is increased. The existence

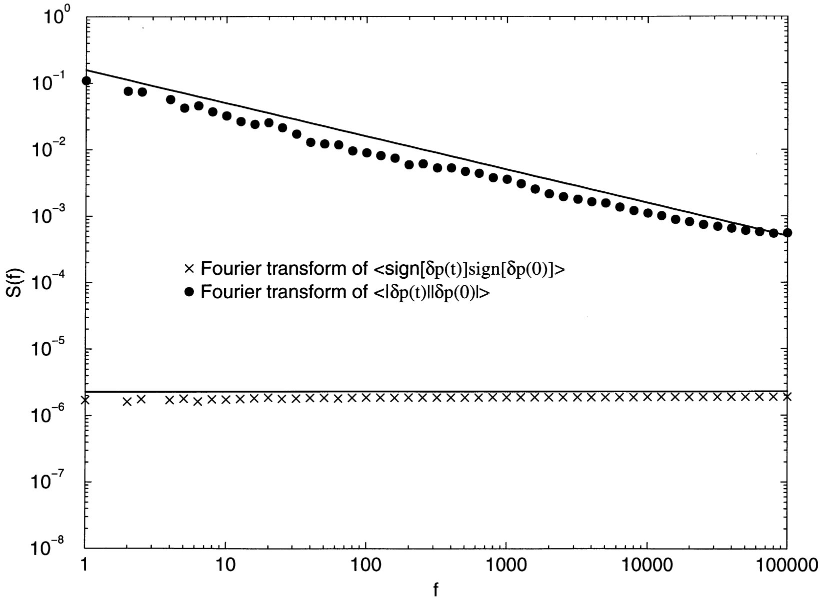

Fig. 2. Fourier transforms of autocorrelation functions of signs of price increments ($\times$) and absolute values of price increments ($\bullet$) averaged over 700 realization of price record $2^{18} \approx 2.6 \times 10^5$ time steps long.

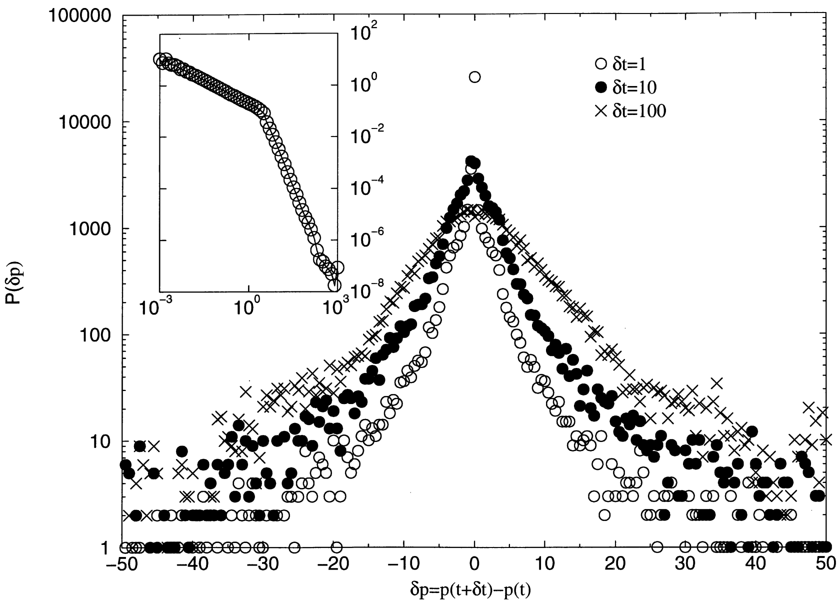

Fig. 3. Histograms of price increments $p(t) = p(t + \delta t) - p(t)$ with time-lags $\delta t = 1$ ($\circ$), 10 ($\bullet$), and 100 ($\times$). The inset shows the histogram of positive price increments $p(t + 1) - p(t) > 0$ (negative increments have a virtually indistinguishable histogram) on a log–log plot. Power law fits in two regions give exponents $1 + \alpha_1 = 0.6$ and $1 + \alpha_2 = 3$.

of a similar power law to power-law crossover was reported in the distribution of stock price increments in NYSE, albeit with different exponents $1+\alpha_1 \simeq 1.4-1.7$, and $1+\alpha_2 \simeq 4-4.5$ [4,5]. The mechanism responsible for this crossover in a real market is at present unclear.

In conclusion, we have introduced and numerically studied a simple model of a limit order-driven market, where agents randomly submit limit or market orders. The execution price of new limit orders is set by offsetting the current market price by a random amount. In spite of such strategy-less, mechanistic behavior of traders, the price time series in our model exhibit a highly non-trivial behavior characterized by long range correlations, fat tails in the histogram of its increments, and a non-trivial Hurst exponent. These results are in qualitative agreement with empirically observed behavior of prices on real stock markets. More work is required to try to modify the rules of our model in order to make this agreement more quantitative.

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